JOURNAL OF COMPUTATIONAL PHYSICS 46, 462–468 (1982)

Note

A Method for the Solution of a Class of Singular Volterra Integro-differential Equations

In this note we describe a procedure which we have found simple to implement, accurate, and efficient for the numerical solution of singular Volterra integro-differential equations of the form

$$\ddot{x} = F(x, \dot{x}, t) + G(x, \dot{x}, t) \int_0^t (t-s)^{-\alpha} \dot{x}(s)\, ds,$$
$$x(0) = a, \qquad \dot{x}(0) = b, \quad 0 \leqslant t \leqslant T, \tag{1}$$

where $\alpha < 1$ and dots denote differentiation. The functions F and G are given and to a large extent arbitrary. Our procedure is particularly efficient when the equation is to be solved over a long interval spanning many orders of magnitude of the variable t. The method and general approach are not restricted to (1) but are readily extendable to other integro-differential equations and also to integral equations.

A very useful feature of our method is the fact that locally, for each t, Eq. (1) is approximated by an ordinary differential equation which can be solved by a marching method. As the integration proceeds the time step is adjusted according to the local magnitude of the derivatives with a very beneficial effect on accuracy and efficiency. In this way the problem of the preselection of a subdivision of the interval $[0, T]$ is completely avoided.

Equations of the type (1) arise in many areas of physics and engineering, typically in phenomena in which the time evolution of the dynamical variable x is governed by diffusive processes depending on x itself. Well-known examples are the motion of a plate in a viscous fluid under the action of external forces [1], problems of heat transfer [2], and surface waves [3, 4]. The present study was motivated by a problem of vapor bubble growth in a superheated liquid. Details of this application can be found in [5].

Under suitable Lipschitz conditions on the functions F and G in (1), it can be shown by an immediate extension of the methods of [6, 7] that Eq. (1) has a unique solution. We shall assume that these conditions are met in the following.

Equations similar to (1), but of first order in the derivative and with x in place of $\dot{x}$ in the integral, have been studied extensively [8–12]. Second-order equations (or, equivalently, first-order systems) have received very little attention, an exception

0021-9991/82/060462-07$02.00/0

being our previous study [13]. Numerical methods for integral equations with the same type of singularity appearing in (1) have also been given [14–16].

To obtain the solution of (1) over a long interval we separate the integral into two parts,

$$\int_0^t (t-s)^{-\alpha}\,\dot{x}(s)\,ds = I_1 + I_2, \tag{2}$$

where

$$I_1 = \int_0^{t_0} (t-s)^{-\alpha}\,\dot{x}(s)\,ds,$$

$$I_2 = \int_{t_0}^{t} (t-s)^{-\alpha}\,\dot{x}(s)\,ds.$$

The point t_0 is chosen such that t_0/t is small in a suitable sense which will be apparent from the following developments. The characteristic feature of our method is the treatment of I_2. Let

$$\eta = 2\,\frac{s-t_0}{t-t_0} - 1.$$

Then the integral I_2 is transformed to

$$I_2 = [\tfrac{1}{2}(t-t_0)]^{1-\alpha} \int_{-1}^{1} (1-\eta)^{-\alpha}\,\dot{x}[t_0 + \tfrac{1}{2}(1+\eta)(t-t_0)]\,d\eta. \tag{3}$$

It is well known that for a definite integral the relation

$$\int_a^b w(x) f(x)\,dx \simeq \sum_{K=1}^{N} c_K f(x_K) \tag{4}$$

(where $a \leqslant x_1 < x_2 < \cdots < x_N \leqslant b$ and w is a given weight function) holds exactly if $f(x)$ is a polynomial of degree $\leqslant (2N-1)$ and the points x_k are chosen as the zeros of the Nth member of the family of polynomials P_l orthogonal in the sense that

$$\int_a^b w(x)\,P_l(x)\,P_m(x)\,dx \propto \delta_{lm}. \tag{5}$$

The constants c_k are referred to as Christoffel numbers and the procedure is known as Gaussian integration [17–19]. In particular, the Jacobi polynomials $P_l^{(\mu,\nu)}$ satisfy (5) with $w(x) = (1-x)^{\mu}\,(1+x)^{\nu}$. Equation (3) is a particular case corresponding to $\mu = -\alpha$, $\nu = 0$.

Unfortunately, $\dot{x}$ in Eq. (3) is, in general, not known at the required zeros of $P_N^{(-\alpha,0)}$. For the computation of this quantity, a third-degree osculating (or Hermite)

interpolation based on the values of x and $\dot{x}$ at the computed points is used [13]. Specifically, if η_K is any one of the zeros of P_N, $s_K = t_0 + \frac{1}{2}(1+\eta_K)(t-t_0)$, and t_i, t_{i+1} are consecutive points at which $\dot{x}, \ddot{x}$ have been computed and such that $t_i < s_K < t_{i+1}$, we set

$$\dot{x}(s_K) \simeq \dot{x}(t_i)\,\psi_i(s_K) + \dot{x}(t_{i+1})\,\psi_{i+1}(s_K) + \ddot{x}(t_i)\,\phi_i(s_K) + \ddot{x}(t_{i+1})\,\phi_{i+1}(s_K). \tag{6}$$

In this equation

$$\psi_i(s) = \left(1 - 2\frac{s-t_i}{t_i - t_{i+1}}\right)\left(\frac{s-t_{i+1}}{t_i - t_{i+1}}\right)^2, \tag{7a}$$

$$\phi_i(s) = (s-t_i)\left(\frac{s-t_{i+1}}{t_i - t_{i+1}}\right)^2, \tag{7b}$$

and ψ_{i+1}, ϕ_{i+1} can be obtained by interchanging the indices i and $i+1$.

Now suppose that the integro-differential equation (1) has been solved up to $t = t_n$. To obtain the value of the solution at $t_{n+1} > t_n$, approximate (1) by an ordinary differential equation in the neighborhood of t_n as follows.

If a sufficiently large value of N is used, the last (or last few) zeros will in general be such that the corresponding value of s is greater than t_n. Let s_M be the last point such that $s_M < t_n$, and let $s_{M+1},\ldots, s_N$ be the remaining points at which $\dot{x}$ is required to compute the integral. Then we write

$$I_2 \simeq \sum_{K=1}^{M} c_K \dot{x}(s_K) + \sum_{L=M+1}^{N} c_L[\dot{x}(t_n)\,\psi_n(s_L) + \dot{x}(t)\,\psi(s_L) + \ddot{x}(t_n)\,\phi_n(s_L) + \ddot{x}(t)\,\phi(s_L)] \tag{8}$$

In this equation $\dot{x}(s_K)$ is to be computed from (6), ψ_n and ϕ_n are given by (7) with t_n and t i place of t_i, t_{i+1}, and ψ, ϕ are again given by (7) with t, t_n in place of t_i, t_{i+1}, respectively. Notice that the values of $\dot{x}$ and $\ddot{x}$ appearing in the second term in (8) do not depend on the summation index so that we may write

$$I_2 \simeq J + A(t)\,\dot{x}(t) + B(t)\,\ddot{x}(t), \tag{9}$$

where

$$J = \sum_{K=1}^{M} c_K \dot{x}(s_K) + \dot{x}(t_n) \sum_{L=M+1}^{N} c_L \psi_n(s_L) + \ddot{x}(t_n) \sum_{L=M+1}^{N} c_L \phi_n(s_L),$$
$$A(t) = \sum_{L=M+1}^{N} c_L \psi(s_L), \qquad B(t) = \sum_{L=M+1}^{N} c_L \phi(s_L). \tag{10}$$

Upon substitution into the original equation (1) we then obtain the following differential approximation to the integro-differential equation:

$$(1 - GB)\,\ddot{x} = F + G(I_1 + J + A\dot{x}). \tag{11}$$

At this point any method suitable for the numerical integration of ordinary differential equations can be used to advance the solution beyond t_n. Notice that a variable step size can be used, which is particularly important if the solution over long intervals is required. Another interesting feature of (11) is the implicit (in principle, although explicit computationally) way in which the contribution of $\ddot{x}(t)$ to the integral is treated. As already remarked, this feature arises only if N is sufficiently large. For small N it may happen that $s_N < t_n$, and consequently $A = B = 0$. The method, however, has also been found to work well in this case.

A final detail concerns the computation of the term I_1. Since t_0/t is assumed small, this term can be computed approximately. The approximation

$$\begin{aligned} I_1 \simeq \frac{t^{2-\alpha}}{(1-\alpha)(2-\alpha)} \Bigg\{ & \left[1 - (2-\alpha)\frac{t_0}{t} + \frac{1}{2}(2-\alpha)(1-\alpha)\left(\frac{t_0}{t}\right)^2\right]\ddot{x}(t_0) \\ & + (2-\alpha)\frac{1}{t}\dot{x}(t_0)\left[1 - (1-\alpha)\frac{t_0}{t}\right] + \frac{(2-\alpha)(1-\alpha)}{t^2}[x(t_0) - x(0)]\Bigg\} \\ & - \frac{(t-t_0)^{1-\alpha}}{1-\alpha}\left[\dot{x}(t_0) + \frac{t-t_0}{2-\alpha}\ddot{x}(t_0)\right] + O\left(\frac{t_0}{t}\right)^3 x^{(3)}(t_0) \end{aligned}$$

obtained by repeated integration by parts performs very satisfactorily for $t_0/t \sim 0.1$. Notice that in this formula only values at $t = 0$ and $t = t_0$ are required. Therefore, all intermediate values can be discarded with considerable saving of storage. For a sufficiently small interval of integration one can take $t_0 = 0$ and, consequently, $I_1 = 0$.

As with many other methods for these types of problems (see, e.g., [8, 15]), to start the computation our procedure requires that the value of the solution be known at the first few points. The method of [13] can be used for this purpose, although frequently asymptotic approximations or simplified equations are available for t small.

To obtain a quantitative idea of the performance of the method, consider the integro-differential equation

$$3x\ddot{x} + \frac{l}{l+1}\dot{x}^2 - \frac{(2l+1)!!}{(2l-2)!!}xt^{-3/2}\int_0^t (t-s)^{-1/2}\dot{x}(s)\,ds = 0,$$

$$x(0) = 0, \qquad \dot{x}(0) = 0$$

where l is a positive integer. The exact solution is $x(t) = t^{l+1}/(l+1)$. The computations were performed with $N = 15$. The initial time step was taken as 0.1, and the initial values were computed from the analytic solution up to $t = 1$. The point t_0 was chosen as the largest value of t for which the solution had been computed such

that $t_0/t \leqslant 0.1$. Equation (11) was solved by a variable-step fourth-order Runge–Kutta method. Values of x and $\dot{x}$ at time $t+h$ were computed in two different ways, first with a single step of length h and, second, with two successive steps of length $h/2$. Denoting the values obtained in the two ways by indices A and B, compute the quantity $E = \max(|x_A/x_B - 1|, |\dot{x}_A/\dot{x}_B - 1|)$. By (repeatedly, if necessary) halving the step size, E was never allowed to exceed 10^{-5}. The step size, however, was doubled if E became less than 10^{-6}. All the computations were performed in *single precision* arithmetic on an IBM 3033 computer. In Table I the computed and analytic results are compared and information on step size and number of steps is provided. The integration was carried on until the overflow limit of the computer was reached. Notice that the relative error gets smaller with increasing l, a trend opposite to what might be expected. Actually, this is an artifact of our step size control. The larger l, the larger the derivatives and consequently the smaller the step size. This can be verified directly by comparing the number of steps and the Δt in Table I.

Extensions of these ideas to other singular (and nonsingular) kernels are trivial. For instance, if $(t-s)^{-\alpha}$ in (1) is replaced b $[s(t-s)]^{1/2}$, $[s(t-s)]^{-1/2}$, instead of the Jacobi polynomials, Chebyshev polynomials of first or second kind should be used [18]. Still other polynomials are available for kernels of the type s^k and $\log(s/t)$ [18]. More complex integrands of the form $f(x, \dot{x}, s)$ in place of $\dot{x}(s)$ can also be treated in a similar way except perhaps for the approximate evaluation of I_1.

An important characteristic of the present method is the osculating approximation (6) of $\dot{x}$. This feature leads to the appearance of $\ddot{x}(t)$ in the approximation (8) to I_2. If the same technique is applied to the first-order equation

$$\dot{x} = F_1(x, t) + G_1(x, t) \int_0^t (t-s)^{-\alpha} x(s)\, ds, \tag{12}$$

I_2 will contain a term $\dot{x}$(t) and a first-order ordinary differential equation approximation analogous to (11) is obtained for (12). The same procedure can also be used for the integral equation

$$x = F_2(x, t) + G_2(x, t) \int_0^t (t-s)^{-\alpha} x(s)\, ds. \tag{13}$$

If $x(s)$ is approximated in terms of x and $\dot{x}$, again a first-order ordinary differential equation is obtained.

This same approach has been used very successfully in [5] to solve a system of the form

$$\ddot{x} = f(x, \dot{x}, y)$$

$$y = \int_0^t (t-s)^{-1/2} \frac{d}{ds} g(x, y)\, ds$$

describing spherical bubble growth in an unbounded liquid. The use of the osculating

TABLE I

	$l=1$			$l=5$			$l=9$		
	Steps	Δt	Error in $\ddot{x}$ (%)	Steps	Δt	Error in $\ddot{x}$ (%)	Steps	Δt	Error in $\ddot{x}$ (%)
3.23×10^{7}	103	6.71×10^{6}	2.9×10^{-3}	211	3.36×10^{6}	1.6×10^{-4}	364	1.67×10^{6}	4.2×10^{-5}
3.33×10^{12}	169	4.40×10^{11}	4.8×10^{-3}	347	2.20×10^{11}	5.0×10^{-4}	—	—	—
5.82×10^{38}	483	8.50×10^{36}	1.4×10^{-2}	—	—	—	—	—	—

approximation (6) on the term dg/ds causes the appearance of $\ddot{y}$ and an approximate second-order ordinary differential equation for y is again obtained. Other generalizations of this method can be found in [13].

ACKNOWLEDGMENTS

This study has been supported by National Science Foundation Grant ENG-78-25497. The author gratefully acknowledges travel support from Gruppo Nazionale di Fisica Matematica of CNR, Italy.

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RECEIVED: July 27, 1981; REVISED: February 10, 1982

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